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Modeling General Relativistic Perfect Fluids in Field-Theoretic Language

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Skew-symmetric massless fields, their potentials being r-forms, are close analogues of Maxwell's field (though the nonlinear cases also should be considered). We observe that only two of them ($r = 2$ and 3) automatically yield stress-energy tensors characteristic of normal perfect fluids. It is shown that they naturally describe both nonrotating ($r = 2$) and rotating general relativistic perfect fluids (then a combination of $r = 2$ and $r = 3$ fields is indispensable) and possess every type of equation of state. Meanwhile, a free $r = 3$ field is completely equivalent to the appearance of the cosmological term in Einstein's equations. Sound waves represent perturbations propagating on the background of the $r = 2$ field. Some exotic properties of these two fields are outlined.

1. INTRODUCTION

Many attempts have been dedicated to giving a translation of (semi-) phenomenological hydrodynamics unto the field-theoretic language (I use the word 'translation' to contrast with the idea of constructing a theory which could automatically give the well-known perfect fluid properties for solutions whose physical meaning is obvious *ab initio*, as well as lead to natural generalizations of old concepts). A thorough review of many publications on the Lagrangian description of general relativistic perfect fluids is given in Brown (1993); practically at the same time a nice paper by Carter (1994) appeared which may be considered as a climax of the era begun by Taub (1954) [probably already even by Clebsch (1859)in Newtonian physics] and later developed by Schutz (1970). One may mention few pages in Hawking and Ellis (1973) on a Lagrangian deduction of the dynamics of perfect fluids, but this subject clearly served there as a secondary accompanying theme

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only. Below I try to avoid the translation-style approach (usually based on introduction of several independent scalar potentials) and consider another way which could be more direct and natural. The translation-like procedure will be used only to illustrate our new approach in concrete examples. Except for a mere mention of the problem of finding new exact solutions of Einstein's equations on the basis of the proposed field-theoretic description of perfect fluids (in the concluding Section 9), I do not touch upon it in this paper.

The main idea is to see what simplest fields automatically possess the form of a stress-energy tensor which is characteristic for a perfect fluid,

$$T^{\mathrm{pf}} = (\mu + p)u \otimes u - pg \tag{1.1}$$

[see a short discussion in Kramer *et al.* (1980), taking into account that we use the metric g with signature $+---$], where p is the invariant pressure of the fluid, μ its invariant mass (energy) density, and u its local four-velocity. We say 'invariant' in the sense that these characteristics are related to the local rest frame of the fluid. The 'simplest' fields are understood as those which are similar in their description to the Maxwell one: they are massless and are described by skew-symmetric potential tensors (of rank r) whose exterior differential represents the corresponding field tensor. Thus the connection coefficients do not enter this description. The Lagrangian densities are functions of quadratic invariants of the field tensors; however, some mixed invariants of the field tensors (and sometimes, potentials) will be used, which should yield the same structure of the stress-energy tensor we need for a perfect fluid (Section 2). When one speaks of a perfect fluid, its isotropy (Pascal's property) and absence of viscosity are necessarily meant. The most characteristic feature of the tensor (1.1) is that it has one single (μ) and one triple ($-p$) eigenvalues in general; this corresponds to Pascal's property. We do not here consider the energy conditions; at least a part of this problem can be "settled" by an appropriate redefinition of the cosmological constant to be then extracted from the stress-energy tensor. Neither shall we consider here thermodynamical properties of fluids; their phenomenological equations of state will be used instead (Kramer *et al.*, 1980), namely the linear equation

$$p = (\gamma - 1)\mu \tag{1.2}$$

and the polytrope one,

$$p = A\mu^{\gamma} \tag{1.3}$$

Applications of these equations of state to nonrotating fluids can be found in Sections 4 and 8 (special relativistic limit) in paragraphs related to equations (4.4), (8.9), and (8.11).

We shall conclude that only ranks $r = 2$ and 3 correspond to (1.1), though only the $r = 2$ case leads to the $\mu + p \neq 0$ term in (1.1), but the fluid is then nonrotating due to the $r = 2$ field equations (Sections 4 and 6); moreover, in this case one comes to a limited class of equations of state. In the pure $r = 3$ case (Section 5), the $u \otimes u$ term in (1.1) is absent ($p = -\mu$), thus reducing the stress-energy tensor to a pure cosmological term, the corresponding field equation naturally yielding $\mu = \text{const}$. The $r = 3$ field, however, proves to be necessary alongside the $r = 2$ one for description of rotating fluids (Section 7), as well as of fluids satisfying more complicated equations of state (*e.g.*, the interior Schwarzschild solution, the end of Section 6). The scalar field case ($r = 0$) does not meet some indispensable requirements and thus should be dropped (Section 3). We give concluding remarks in Section 9.

2. STRESS-ENERGY TENSOR

It is well known that when the action integral of a physical system is invariant under general transformations of the space–time coordinates, the (second) Noether theorem yields definitions and conservation laws of a set of dynamical characteristics of the system. These are, in particular, its (symmetric) stress-energy tensor and (canonical) energy-momentum pseudotensor. The latter is important in establishing the commutation relations for the creation and annihilation operators (the second-quantization procedure), while the former acts as the source term in Einstein's field equations. Both objects are mutually connected by the Belinfante–Rosenfeld relation. This paper focuses on a study of the stress-energy tensor of the rank-2 and rank-3 fields described by skew-symmetric tensor potentials (2- and 3-forms) whose exterior differentials serve as the corresponding field strengths. As already mentioned, this approach does not involve the Christoffel symbols when these fields and their interaction with gravitation are described in a coordinated basis, thus representing the simplest scheme which resembles the general relativistic theory of the electromagnetic field.

It is worth recalling some general definitions and relations leading to the stress-energy tensor. Under an infinitesimal coordinate transformation $x'^{\mu} = x^{\mu} + \epsilon\xi^{\mu}(x)$, components of a tensor or tensor density change as

$$\delta A_a := A'_a(x') - A_a(x) =: \epsilon A_a|^{\tau}_{\sigma}\xi^{\sigma}_{,\tau}$$

(up to the first-order terms; this law is, naturally, the definition of $A_a|^{\tau}_{\sigma}$), a being a collective index [the notations of Trautman (1956), sometimes used in the formulation of the Noether theorem and the general description of the covariant derivative of arbitrary tensors and tensor densities in Riemannian geometry: $A_{a;\alpha} = A_{a,a} + A_a|^{\tau}_{\sigma}\Gamma^{\sigma}_{\alpha\tau}$). Then the Lie derivative of A_a with respect to a vector field ξ takes the form

$$\mathcal{L}_\xi A_a = A_{a,\sigma}\xi^\sigma - A_a|^\tau_\sigma \xi^\sigma_{,\tau} \equiv A_{a;\sigma}\xi^\sigma - A_a|^\tau_\sigma \xi^\sigma_{;\tau} \tag{2.1}$$

The stress-energy tensor density corresponding to a Lagrangian density $\mathfrak{L}$ follows from the Noether theorem (Noether, 1918; Mitskievich, 1958; Mitskievich, 1969) as

$$\mathfrak{T}^\beta_\alpha := \frac{\delta\mathfrak{L}}{\delta g_{\mu\nu}} g_{\mu\nu}|^\beta_\alpha \equiv \frac{\delta\mathfrak{L}}{\delta g^{\mu\nu}} g^{\mu\nu}|^\beta_\alpha \tag{2.2}$$

Usually a rank-two tensor, and not its density, is considered,

$$T^\beta_\alpha = (-g)^{-1/2}\mathfrak{T}^\beta_\alpha, \qquad T^\beta_{\alpha;\beta} = 0 \tag{2.3}$$

Turning now to fields with skew-symmetric potentials, one has for a rank-r tensor field

$$F_{\mu\alpha\ldots\beta} := (r+1)A_{[\alpha\ldots\beta,\mu]} \equiv (r+1)A_{[\alpha\ldots\beta;\mu]} \tag{2.4}$$

where the field potential A and the field tensor $F = dA$ are covariant skew-symmetric tensors of ranks r and $r + 1$, respectively while

$$A = \frac{1}{r!} A_{\alpha\ldots\beta}\, dx^\alpha \wedge \ldots \wedge dx^\beta, \qquad F = \frac{1}{(r+1)!} F_{\mu\alpha\ldots\beta}\, dx^\mu \wedge dx^\alpha \wedge \ldots \wedge dx^\beta$$

The quadratic invariant of the field tensor is

$$I = *(F \wedge *F) \equiv -\frac{1}{(r+1)!} F_{\alpha_1\ldots\alpha_{r+1}} F_{\beta_1\ldots\beta_{r+1}} g^{\alpha_1\beta_1} \ldots g^{\alpha_{r+1}\beta_{r+1}} \tag{2.5}$$

[An obvious special case is the electromagnetic (Maxwell) field ($r = 1$). From the expression (2.9) on, we shall use the notations A and F for the potential and field tensor forms of the electromagnetic, or $r = 1$, field only, as well as I for the corresponding invariant.]

Lagrangian densities of the fields under consideration will be taken in the general form $\mathfrak{L} = \sqrt{-g}L(I)$, $L(I)$ being a scalar algebraic function of the invariant (2.5). Then relations (2.2) and (2.3) yield

$$T^\beta_\alpha = -L\delta^\beta_\alpha - 2\frac{\partial L}{\partial g_{\mu\beta}} g_{\mu\alpha} \equiv -L\delta^\beta_\alpha + 2\frac{\partial L}{\partial g^{\mu\alpha}} g^{\mu\beta} \tag{2.6}$$

so that, since L depends on the metric tensor only via I and due to (2.5),

$$T^\beta_\alpha = -L\delta^\beta_\alpha - \frac{2}{s!}\frac{dL}{dI} F_{\alpha\mu_1\ldots\mu_s} F^{\beta\mu_1\ldots\mu_s} \tag{2.7}$$

It is easy to see that field equations can be similarly rewritten using the function $L(I)$:

$$\frac{\partial \mathfrak{L}}{\delta A_{\alpha\ldots\beta}} := \frac{\partial \mathfrak{L}}{\partial A_{\alpha\ldots\beta}} - \left(\frac{\partial \mathfrak{L}}{\partial A_{\alpha\ldots\beta,\mu}}\right)_{,\mu} = 0 \Rightarrow \left(\sqrt{-g}\,\frac{dL}{dI}\,F^{\alpha\ldots\beta\mu}\right)_{,\mu} = 0 \quad (2.8)$$

Further, a more general Lagrangian density is worth considering,

$$\mathfrak{L} = \sqrt{-g}L(H, I, J, K) \quad (2.9)$$

a function of invariants of (skew-symmetric) fields of ranks 0, 1, 2, and 3:

$$\left.\begin{array}{l} H = *(d\varphi \wedge *d\varphi) = -\varphi_{,\alpha}\varphi^{,\alpha} \\ I = *dA \wedge *dA) = -(1/2)F_{\mu\omega}F^{\mu\omega}, \qquad F = dA \\ J = *(dB \wedge *dB) = -(1/3!)G_{\lambda\mu\omega}G^{\lambda\mu\omega} = \tilde{G}_{\kappa}\tilde{G}^{\kappa}, \\ G = dB, \quad B*^{\mu\nu}{}_{;\nu} = -\tilde{G}^{\mu} \\ K = *(dC \wedge *dC) = -(1/4!)W_{\kappa\lambda\mu\nu}W^{\kappa\lambda\mu\nu} = \tilde{W}^2, \qquad W = dC \end{array}\right\} \quad (2.10)$$

where $*$ before an object is the Hodge star, and the duality relations hold:

$$B\overset{\mu\nu}{*} = \frac{1}{2}E^{\mu\nu\alpha\beta}B_{\alpha\beta}, \qquad G_{\lambda\mu\nu} = \tilde{G}^{\kappa}E_{\kappa\lambda\mu\nu}, \qquad W_{\kappa\lambda\mu\nu} = \tilde{W}\,E_{\kappa\lambda\mu\nu} \quad (2.11)$$

$E_{\kappa\lambda\mu\nu} = \sqrt{-g}\epsilon_{\kappa\lambda\mu\nu}$ is the Levi-Civita skew-symmetric axial tensor, while $\epsilon_{0123} = +1$. Here p-forms are defined with respect to a coordinated basis as

$$f = (1/p!)f_{\nu_1\nu_2\ldots\nu_p}\,dx_1^{\nu} \wedge dx_2^{\nu} \wedge \ldots \wedge dx_p^{\nu}$$

As an obvious generalization of (2.2) and hence of (2.6), the stress-energy tensor corresponding to (2.9) then takes the form

$$T^{\beta}_{\alpha} = -L\delta^{\beta}_{\alpha} - 2\frac{\partial L}{\partial H}\varphi_{,\alpha}\varphi^{,\beta} - 2\frac{\partial L}{\partial I}F_{\alpha\mu}F^{\beta\mu}$$

$$+ 2J\frac{\partial L}{\partial J}(\delta^{\beta}_{\alpha} - u_{\alpha}u^{\beta}) + 2K\frac{\partial L}{\partial K}\delta^{\beta}_{\alpha} \quad (2.12)$$

where $u_{\alpha} = \tilde{G}_{\alpha}/J^{1/2}$. When $u \cdot u = 1$, the real vector u is timelike, and if imaginary, it corresponds then to a spacelike real vector. We do not consider here the null vector case ($u \cdot u = 0$).

The expressions (2.6), (2.7), and (2.12) are equivalent to those which involve variational derivatives with respect to the metric tensor, (2.2), if the Lagrangian density is considered as a function of the quadratic invariants H, I, J, and K.

3. FREE (IN GENERAL, NONLINEAR) SCALAR FIELD

In the free scalar field case, $\mathfrak{L} = \sqrt{-g}L(H)$, one could also consider the (normalized) gradient of the scalar field potential φ as another four-velocity (say $\overset{0}{u}_\alpha = \varphi_{,\alpha}/\sqrt{|H|}$), but this vector obviously can be timelike only if the scalar field is essentially nonstationary (as to the four-velocity u due to the 2-form field B, the vector $\tilde{G}$ is automatically timelike for stationary or static fields). In fact, the t dependence should *dominate* in φ, and this means that for scalar fields normal and abnormal fluids exchange their roles (see the next section, where these concepts are also discussed).

For the sake of completeness, we mention here the field equation

$$\left(\sqrt{-g}\,\frac{dL}{dH}\,\varphi^{,\alpha}\right)_{,\alpha} = 0 \tag{3.1}$$

and the stress-energy tensor

$$T^\beta_\alpha = -L\delta^\beta_\alpha - 2\,\frac{\partial L}{\partial H}\,\varphi_{,\alpha}\varphi^{,\beta} \tag{3.2}$$

of a free massless scalar field. T^β_α has then one single and one triple eigenvalues, which we denote, as was done for perfect fluids in (1.1), as μ and $-p$, respectively,

$$\mu = 2H\,\frac{dL}{dH} - L, \qquad p = L \tag{3.3}$$

From these expressions we see that, if some incoherent fluid (dust) would be described by this field, the Lagrangian L should vanish, so that the invariant H has to be (at least) constant for this solution. But then the mass density becomes constant, too, this description being obviously applicable only to completely unphysical dust distributions.

These observations clearly show that the scalar field has to be excluded from the list of fields suitable for the description of normal perfect fluids.

4. FREE RANK-2 FIELD

Let us next consider a free rank-2 field (L being a function only of J); thus the stress-energy tensor (2.12) reduces to

$$T^\beta_\alpha = \left(2J\,\frac{dL}{dJ} - L\right)\delta^\beta_\alpha - 2J\,\frac{dL}{dJ}\,u_\alpha u^\beta \tag{4.1}$$

Here, u evidently is eigenvector of the stress-energy tensor:

$$T^\beta_\alpha u^\alpha = -Lu^\beta$$

while any vector orthogonal to u is also eigenvector, this time with the (triple) eigenvalue $2J\,dL/dJ - L$. This is exactly the property of the stress-energy tensor of a perfect fluid, the only additional condition being that the vector u should be a real timelike one. The latter depends, however, on the concrete choice of solution of the rank-3 field equations. Thus we come to the conclusion that

$$\mu = -L \qquad \text{and} \qquad p = L - 2J\frac{dL}{dJ} \tag{4.2}$$

μ is the invariant mass density and p is the pressure of the fluid. One may, of course, reinterpret this tensor as a sum of the stress-energy tensor proper and (in general) a cosmological term.

The free field equations for the field tensor G reduce to

$$\left(J^{1/2}\frac{dL}{dJ}u_\kappa\right)_{,\lambda} = \left(J^{1/2}\frac{dL}{dJ}u_\lambda\right)_{,\kappa} \Rightarrow J^{1/2}\frac{dL}{dJ}u_\lambda \equiv \frac{dL}{dJ}\tilde{G}_\lambda = \Phi_{,\lambda} \tag{4.3}$$

$u \cdot u = 1$ by the definition. Thus the free $r = 2$ field case can describe nonrotating fluids only, since the vector field u (or, equivalently, $\tilde{G}$) determines a nonrotating congruence. In order to identify u with the fluid's four-velocity, one has to consider solutions with u real and timelike (we call this the normal fluid case). The null case was already excluded from consideration, and when $\tilde{G}$ is spacelike, one may interpret the corresponding solution as describing a tachyonic (abnormal) fluid. The latter notion seems to be somewhat odious, but it should be introduced if one formulates a classification of all possible cases of perfect fluid-like stress-energy tensors (the well-known energy conditions are closely related to this subject). We do not consider the tachyonic fluid case below; moreover, we shall now show that all static, spherically symmetric solutions of the rank-2 skew-symmetric field equations automatically yield timelike vector field $\tilde{G}$; this should be only a part of a larger family of physically acceptable solutions.

Perfect fluids characterized by (1.2) correspond to a homogeneous function of J as the Lagrangian, $L = -\sigma J^{\gamma/2}$, $\sigma > 0$. The important special cases are then the incoherent dust ($p = 0$) for $\gamma = 1$, incoherent radiation ($p = \mu/3$) for $\gamma = 4/3$, and stiff matter ($p = \mu$) for $\gamma = 2$.

One may similarly treat polytropes, (1.3), though in this case the Lagrangian is determined only implicitly. We introduce here a notation $L = -\lambda(J)$; then $\mu + p = \lambda + A\lambda^\gamma = 2J\, d\lambda/dJ$ and

$$J = \exp\left[2\int\frac{d\lambda}{\lambda + A\lambda^\gamma}\right] \tag{4.4}$$

where A and γ are considered as constants. It is clear what kind of difficulty one has to confront now: even approximately, this relation cannot be resolved with respect to λ, though, of course, polytropic fluids are well described in the field-theoretic language after all. A possibility to write some function explicitly is a mere convenience and not a necessity.

One could begin the formulation of this approach with phenomenological consideration of a perfect fluid[2] just postulating the form of its stress-energy tensor (1.1) and taking a general equation of state in the form $\mu = \mu(p)$. Define

$$\rho = \exp\left[\int \frac{d\mu/dp}{\mu + p}\, dp\right] \tag{4.5}$$

Then the conservation $T^{\mu\nu}{}_{;\nu} = 0$ implies $(\rho u^{\nu})_{;\nu} = 0$. Therefore, a skew-symmetric tensor (superpotential) $\tilde{B}^{\mu\nu}$ should exist such that $\rho u^{\mu} = \tilde{B}^{\mu\nu}{}_{;\nu}$. A direct comparison of (4.5) and (4.2) shows that $\rho = J^{1/2}$, since, denoting $\tilde{B}^{\mu\nu}{}_{;\nu}$ as $\tilde{G}^{\mu}$, we see that $J = \tilde{G}\cdot\tilde{G}; = \rho^2$ [cf. the notations in (2.10)]. This shows that it is only natural to use a rank-2 field for description of a perfect fluid, and the invariant J is automatically suggested; however, this heuristic approach is more closely related to the case of a Lagrangian only linearly depending on J.

In the static, spherically symmetric case, with a diagonal metric in the curvature coordinates, one has to choose

$$B = \sin\vartheta\, A(r)\, d\vartheta \wedge d\phi, \qquad G = \sin\vartheta\, A'(r)\, dr \wedge d\vartheta \wedge d\phi$$

where the function $\sin\vartheta$ appears to make the stress-energy tensor dependent only on the radial coordinate; the standard spherical coordinate notations are used. In this case,

$$J = -A'^2 \sin^2\vartheta\, g^{rr} g^{\vartheta\vartheta} g^{\phi\phi} > 0$$

For a natural 1-form basis comoving with the fluid,

$$\theta^{(0)} = \sqrt{g00\ dt} = u, \qquad \theta^{(1)} = \sqrt{-g_{rr}\ dr},$$
$$\theta^{(2)} = r\, d\vartheta, \qquad \theta^{(3)} = r\sin\vartheta\, d\phi$$

the stress-energy tensor reads

$$T = -L(J)\theta^{(0)} \otimes \theta^{(0)} + \left(2J\frac{dL}{dJ} - L\right)$$
$$\times (\theta^{(1)} \otimes \theta^{(1)} + \theta^{(2)} \otimes \theta^{(2)} + \theta^{(3)} \otimes \theta^{(3)})$$

in conformity with (4.2).

[2] The idea was suggested by J. Ehlers.

5. FREE RANK-3 FIELD. THE ONLY INTERPRETATION: COSMOLOGICAL TERM

In this case the Lagrangian depends only on the invariant K; thus

$$T^{\beta}_{\alpha} = \left(2K\frac{dL}{dK} - L\right)\delta^{\beta}_{\alpha} = -\frac{\Lambda}{\kappa}\,\delta^{\beta}_{\alpha} \tag{5.1}$$

κ is Einstein's gravitational constant. This stress-energy tensor is merely proportional to the metric tensor; therefore the coefficient $2KdL/dK - L = -\Lambda/\kappa$ obviously should be constant. It is trivially constant (and equal to zero) indeed when $L \sim K^{1/2}$, the field components $W^{\kappa\lambda\mu\nu}$ being then arbitrary. Otherwise, it becomes constant (and nonzero) due to the field equations to which vanishing of the stress-energy tensor divergence is equivalent. Indeed, the equations

$$\left(\sqrt{-g}\,\frac{dL}{dK}\,W^{\kappa\lambda\mu\nu}\right)_{,\nu} = 0 \tag{5.2}$$

reduce to

$$K^{1/2}\,\frac{dL}{dK} = \text{const} \tag{5.3}$$

since $\sqrt{-g}E^{\kappa\lambda\mu\nu} = -\epsilon_{\kappa\lambda\mu\nu} = \text{const}$. We see that both cases (when $L \sim K^{1/2}$ and $L \nsim K^{1/2}$) exactly correspond to the above conclusions. In the first case this does not need comment, but when $L \nsim K^{1/2}$, the left-hand-side expression in (5.3) is really a function of K. Hence from (5.3) it follows that K itself should be constant. Thus the 'cosmological constant' Λ which appears in (5.1) is really constant due to the field equations. These equations, in a sharp contrast to the usual equations of mathematical physics, cannot be characterized as hyperbolic ones (or otherwise). Moreover, the case of $L \sim K^{1/2}$ corresponds to vanishing of the cosmological constant, and the field equations now impose no conditions on K whatsoever—the rank-3 field is then *arbitrary* due to the field equation, a very particular situation for the field theory indeed!

If $L = \sigma K^k$ with a positive constant σ, then $2k < 1$ corresponds to the de Sitter case; $2k = 1$, to the absence of cosmological constant (this is the case of a *phantom* rank-3 field which is completely arbitrary, and otherwise does not produce any stress-energy tensor at all); finally, $2k > 1$ corresponds to the anti-de Sitter case [see for standard definitions Hawking and Ellis, (1973)]. We propose to call the rank-3 field a cosmological field; another—Machian—reason for this will become obvious after a consideration of rotating fluids.

6. NONROTATING FLUIDS

In a comoving frame, the local four-velocity of a fluid is $u^\mu \sim \delta_0^\mu$, and the x^0 coordinate lines should form a nonrotating congruence. Since $u \cdot u = 1$, in the case of a normal fluid,

$$u^\mu = \delta_0^\mu / \sqrt{g_{00}}, \qquad \tilde{G}^\mu = \Xi \delta_0^\mu \tag{6.1}$$

with Ξ being a function of the four (in general) coordinates. Thus

$$J = \Xi^2 g_{00} \qquad \text{and} \qquad u^\mu = \tilde{G}^\mu / \sqrt{J} \tag{6.2}$$

To be more concise, we shall consider here the case of a homogeneous function $L(J) = \sigma J^k$. Then $J^{k-1} \tilde{G}_\lambda = \tilde{\Phi}_{,\lambda}$, $\tilde{\Phi}$ being a pseudopotential (with the pseudoscalar property).

Let us now consider some perfect fluid solutions in general relativity [for excellent reviews see Kramer *et al.* (1980) and Delgaty and Lake (1998)]. It is convenient to write this solution in comoving coordinates. Moreover, let the fluid satisfy an equation of state $p = (2k - 1)\mu$ with $k = \text{const}$. Apart from the metric coefficients, there will be only one independent function characterizing the fluid (and its motion), say, μ. In the scheme outlined above, this function should be related to Ξ, the only independent function involved in the $r = 2$ field (the metric tensor is supposed to be the same in the perfect fluid and $r = 2$ field languages). Clearly, the problem then reduces to a determination of the relationship between the two functions. One finds immediately

$$\mu = \sigma J^k; \qquad \text{thus} \quad \Xi = (\mu/\sigma)^{1/2k} / \sqrt{g_{00}} \tag{6.3}$$

Hence,

$$\tilde{G}^\mu = (1/\sqrt{g_{00}})\, (\mu/\sigma)^{1/2k} \delta_0^\mu, \qquad \tilde{\Phi}_{,\mu} = k(\mu/\sigma)^{(2k-1)/2k} g_{0\mu} / \sqrt{g_{00}} \tag{6.4}$$

Example. The Klein Metric. The Klein metric (Klein, 1947; Kramer *et al.*, 1980) describes a static space–time filled with incoherent radiation, $p = \mu/3$. In this case,

$$ds^2 = r\, dt^2 - \frac{7}{4}\, dr^2 - r^2(d\vartheta^2 + \sin^2 \vartheta\, d\phi^2)$$

$$\mu = 3/(7\aleph r^2), \qquad k = 2/3$$

Then, obviously,

$$\Xi = \left(\frac{3}{7\kappa\sigma}\right)^{3/4} \frac{1}{r^2}, \qquad \tilde{\Phi} = \frac{2}{3} \left(\frac{3}{7\kappa\sigma}\right)^{1/4} t$$

Example. The Tolman–Bondi Solution (Tolman, 1934; Bondi, 1947). Now,

$$ds^2 = d\tau^2 - \exp[\lambda(\tau, R)]\, dR^2 - r^2(\tau, R)(d\vartheta^2 + \sin^2 \vartheta\, d\phi^2)$$

$$\mu = \frac{F'}{\kappa r' r^2}, \qquad p = 0, \quad k = 1/2$$

$$r = \frac{F}{2f}(\cosh \eta - 1), \qquad \sinh \eta - \eta = \frac{2f^{3/2}}{F}(\tau_0 - \tau)$$

F, f, and τ_0 are arbitrary functions of R. The translation into the $s = 2$ field language reads simply

$$\Xi = \frac{F'}{\kappa\sigma r' r^2}, \qquad \tilde{\Phi} = \tau.$$

It is equally easy to cast the Friedmann–Robertson–Walker cosmological solutions in the rank-2 field form (in fact, the FRW universe filled with an incoherent dust represents a special case of the Tolman–Bondi solution).

Example. The Interior Schwarzschild Solution (Kramer *et al.*, 1980). The interior Schwarzschild solution is now a special case to be treated in more detail. Its characteristic feature is that the mass density of the fluid with which it is filled is constant, while the fluid's pressure decreases when the radial coordinate grows, vanishing on some spherical boundary (thus making it possible to join this solution with the exterior vacuum region). However, this property clearly contradicts the relation between μ and p obtainable from a Lagrangian depending on one invariant, J, only. Therefore one has to consider interaction, say, of $r = 2$ and $r = 3$ fields. We choose the corresponding Lagrangian to be $L(J, K) = -M(J)(1 - \alpha K^{1/2})$ (the rank-3 field obviously being a phantom one). Then

$$T^{\beta}_{\alpha} = \left[M(J) - 2J\frac{dM}{dJ}(1 - \alpha K^{1/2}) \right]\delta^{\beta}_{\alpha} + 2J\frac{dM}{dJ}(1 - \alpha K^{1/2})u_{\alpha}u^{\beta}$$

Hence the former expression for μ is not changed, but pressure is now a function of the invariant K *arbitrarily* depending on coordinates:

$$\mu = M(J), \qquad p = 2J\frac{dM}{dJ}(1 - \alpha K^{1/2}) - M(J) \tag{6.5}$$

The fact that K really may be chosen arbitrarily follows from the field equations. For the $r = 2$ field one has

$$d\left[\frac{dM}{dJ}(1 - \alpha K^{1/2})\tilde{G}\right] = 0 \tag{6.6}$$

and for the $r = 3$ field,

$$M(J) = \text{const} \tag{6.7}$$

without any other conditions on K. The latter equation is exactly what we needed, and the first one then reduces to $d\,[(1 - \alpha K^{1/2})\tilde{G}] = 0$ or, in the static case when K is independent of x^0 and $\tilde{G} = J^{1/2}\sqrt{g_{00}}\,dx^0$, simply to

$$(1 - \alpha K^{1/2})\sqrt{g_{00}} = q^2 \tag{6.8}$$

where q is a constant.

However, the last equation seems to impose a critically strong restriction on the choice of K (yet having been arbitrary) which should now automatically fit the expression for pressure. Let us see if this is the case for the interior Schwarzschild solution. The latter is described by

$$\left.\begin{aligned} ds^2 &= \left(a - b\sqrt{1 - \frac{r^2}{R^2}}\right)^2 dt^2 - \frac{dr^2}{1 - r^2/R^2} - r^2(d\vartheta^2 + \sin^2\vartheta\, d\phi^2) \\ \mu &= \frac{3}{\kappa R^2}, \qquad p = \frac{3}{\kappa R^2}\left(\frac{2a}{3\sqrt{g_{00}}} - 1\right) \end{aligned}\right\} \tag{6.9}$$

a, b, and R are constants (for details see Kramer *et al.* (1980). If we take $M = \sigma J^k$, it is readily found that all conditions are satisfied indeed for $k = a/3q$. Then for $\mu = \text{const}$ it is always possible to consider a linear $r = 2$ field, $k = 1$: one has only to choose $q = a/3$.

7. ROTATING FLUIDS

We have come to the conclusion that the $r = 2$ and $r = 3$ fields have stress-energy tensors possessing eigenvalues typical of perfect fluids: in the free field case, the $r = 2$ field with eigenvalues characteristic of a usual isotropic perfect fluid, and the $r = 3$ field with only one quadruple eigenvalue (thus the stress-energy tensor is proportional to the metric tensor: the cosmological term form). For description of a perfect fluid with the equation of state $p = (\gamma - 1)\mu$ and a given constant value of γ one needs only one function, say, the mass density μ [the metric tensor is considered as already given, and the system of coordinates is supposed to be comoving with the fluid, thus the four-velocity vector is $u^\mu = (g_{00})^{-1/2}\delta^\mu_0)$]. It seemed that this situation in all cases fits well for translating into the $r = 2$ field language. But we were confronted with the no-rotation condition for a perfect fluid when the rank-2 field was considered to be free. It is clear that the only remedy in this case is to introduce a nontrivial source term in the $r = 2$ field equations, thus a change to the non-free-field case or, at least, to include in the Lagrangian a dependence on the rank-2 field potential B.

The simplest way to do this is to introduce in the Lagrangian density a dependence on a new invariant $J_1 = -B_{[\kappa\lambda}B_{\mu\nu]}B^{[\kappa\lambda}B^{\mu\nu]}$ which does not spoil the structure of the stress-energy tensor, simultaneously yielding a source term (thus enabling us to eliminate the no-rotation property) without changing the divergence term in the $r = 2$ field equations. We shall use below three invariants: the obvious ones, J and K, and the invariant just introduced of the $r = 2$ field *potential*, J_1. One easily finds that

$$B_{[\kappa\lambda}B_{\mu\nu]} = -\frac{2}{4!}\, B_{\alpha\beta}\overset{\alpha\beta}{B*}\, E_{\kappa\lambda\mu\nu} \tag{7.1}$$

where $\overset{\alpha\beta}{B*} := \frac{1}{2} B_{\mu\nu}E^{\alpha\beta\mu\nu}$ (dual conjugation). Thus $J_1^{1/2} = 6^{-1/2}B_{\alpha\beta}\overset{\alpha\beta}{B*}$. In fact, $J_1 = 0$, if B is a simple bivector ($B = a \wedge b$, a and b being 1-forms; only the four-dimensional case to be considered); this corresponds to all types of rotating fluids discussed in the literature. This *cannot, however, annul* the expression which this invariant contributes to the $r = 2$ field equations: up to a factor, it is equal to $\partial J_1^{1/2}/\partial B_{\mu\nu} \neq 0$. Thus let the Lagrangian density be

$$\mathfrak{L} = \sqrt{-g}(L(J) + M(K)J_1^{1/2}) \tag{7.2}$$

The $r = 2$ field equations now take the form [*cf.* (4.3)]

$$d\left(\frac{dL}{dJ}\,\tilde{G}\right) = \sqrt{2/3}M(K)B \tag{7.3}$$

which means that introduction of rotation of the fluid destroys the gauge freedom of the $r = 2$ field. In turn, the $r = 3$ field equations [*cf.* (5.2) and (5.3)] yield the first integral

$$J_1^{1/2}K^{1/2}\,\frac{dM}{dK} = \text{const} \equiv 0 \tag{7.4}$$

(when $J_1 = 0$, as just stated). It is obvious that K (hence, M) *arbitrarily* depends on the space–time coordinates if only the $r = 3$ field equations are taken into account. Though the $r = 2$ field equations (7.3) apparently show that the $\tilde{G}$ congruence should in general be rotating, the $r = 2$ field B is an exact form for solutions with constant M (K), thus its substitution into the left-hand side of (7.3) via $\tilde{G}$ leads trivially to vanishing of G (and hence B). Hence in a nontrivial situation the cosmological field K [see (2.10)] has to be essentially nonconstant.

But the complete set of equations contains Einstein's equations as well. One has to consider their sources and the structure of their solutions (some of which fortunately are available) in order to better understand this remarkable situation probably never encountered in theoretical physics before.

The stress-energy tensor which corresponds to the new Lagrangian density (7.2) is

$$T_\alpha^\beta = \left(-L - MN + 2J\frac{dL}{dJ} + 2KN\frac{dM}{dK} + 2J_1 M\frac{dN}{dJ_1} \right)\delta_\alpha^\beta - 2J\frac{dL}{dJ}\,u_\alpha u^\beta \tag{7.5}$$

where we have used $N(J_1) = J_1^{1/2}$. It is obvious that only the terms involving L and J survive here ($J_1 = 0 = N$). For a perfect fluid with the equation of state $p = (\gamma - 1)\mu$, one finds $L = -\sigma J^{\gamma/2}$, thus $T_\alpha^\beta = -\gamma L u_\alpha u^\beta + (\gamma - 1)L\delta_\alpha^\beta$.

Then one has a translation algorithm between the traditional perfect fluid and $r = 2$ field languages:

$$\left.\begin{aligned} &\mu = -L = \sigma J^{\gamma/2}, \qquad \tilde{G}^\mu = \Xi\delta_t^\mu, \qquad \Xi = \frac{1}{\sqrt{g_{00}}}\left(\frac{\mu}{\sigma}\right)^{1/\gamma} \\ &G = dB = \sqrt{3/2d}\left(\frac{1}{M(K}\right)\wedge d\left(\frac{dL}{dJ}\,\tilde{G}\right) \end{aligned}\right\} \tag{7.6}$$

[*cf.* (7.3)]. The function M depends arbitrarily on coordinates; thus one can choose its appropriate form using the last relation without coming into contradiction with the dynamical equations.

We see that the cosmological field K plays a very special role in the description of rotating fluids. This field makes it possible to consider rotation, but its own field equations do not impose any restriction on K. (A similar situation, but without rotation, was observed above in the case of the interior Schwarzschild solution.) In each case, one has to adjust the K field using the gravitational field solutions, thus from global considerations (this being the final analysis of considerations of the last paragraphs). Together with the fact that the free K field results in the introduction of the cosmological constant, these properties of the cosmological field recall the ideas of the Mach principle and a practically forgotten hypothesis due to Sakurai (1960).

Example. The Gödel Universe (Gödel, 1949). Gödel's universe filled with a rotating perfect fluid is described by

$$ds^2 = a^2\,(dt^2 + 2\sqrt{2}z\;dt\;dx + z^2\;dx - dy^2 - z^{-2}\;dz^2), \qquad \sqrt{-g} = a^4$$

$$p = \mu = \frac{1}{2\kappa a^2}, \qquad u^\mu = a^{-1}\delta_t^\mu, \qquad k = 1 \quad (\gamma = 2)$$

[in Kramer *et al.* (1980) p and μ take other values, since a cosmological term is considered there, but this is only a matter of convention; moreover, there are misprints in the book: the factor a^2 should be put in the denominator, as we have written above]. Now it is easy to find

$$\Xi = (2\kappa\sigma a^4)^{-1/2}, \qquad \tilde{G} = (2\kappa\sigma)^{-1/2}\,(dt + \sqrt{2}z\,dx), \qquad J = (2\kappa\sigma a^2)^{-1}$$

while $G = a^2(2\kappa\sigma)^{-1/2}\,dx \wedge dy \wedge dz$. Hence [see also (7.3)]

$$d\left(\frac{dL}{dJ}\tilde{G}\right) = \sqrt{\sigma/\kappa}\,dx \wedge dz = \sqrt{2/3}M(K)B$$

so that

$$G = dB = \sqrt{\frac{3\sigma}{2\kappa}}\,d\left(\frac{1}{M}\right) \wedge dx \wedge dz$$

This gives

$$M = -\frac{\sqrt{3\sigma}}{a^2 y} \qquad \text{and} \qquad B = \frac{a^2 y}{\sqrt{2\kappa\sigma}}\,dx \wedge dz$$

Example. Davidson's Fluid (Davidson, 1996). Another stationary solution with fluid being in a certain sense in a rigid-body rotation is described by the metric

$$ds^2\ P(dt + \sqrt{23/8}\,ar^2 d\phi)^2 - r^2P^3 d\sigma^2 - P^{-3/4}(dr^2 + dz^2)$$

$\sqrt{-g} = rP^{5/4}$, while

$$P = \sqrt{1 + a^2r^2}, \qquad \gamma = 5/3, \qquad \mu = (9a^2/2\kappa)P^{-5/4}$$

We find

$$\Xi = \left(\frac{9a^2}{2\kappa\sigma}\right)^{3/5} P^{-5/4}, \qquad J = \left(\frac{9a^2}{2\kappa\sigma}\right)^{6/5} P^{-3/2}$$

$$\frac{dL}{dJ}\tilde{G} = -\frac{5\sigma}{6}\left(\frac{9a^2}{2\kappa\sigma}\right)^{2/5}(dt + \sqrt{23/8}ar^2 d\phi),$$

$$G = \left(\frac{9a^2}{2\kappa\sigma}\right)^{3/5} r\,dr \wedge dz \wedge d\phi$$

$$M = \left(\frac{9a^2}{2\kappa\sigma}\right)^{-1/5}\frac{5\sqrt{23}\sigma a}{4\sqrt{3}z}, \qquad B = -\left(\frac{9a^2}{2\kappa\sigma}\right)^{3/5} zr\,dr \wedge d\phi$$

In both of these examples we have determined M as a function of a coordinate without mentioning the $r = 3$ field tensor, since in the rotating perfect fluid theory only the coordinate dependence of M matters. It is clear that our considerations are in complete agreement with the field equations.

8. SPECIAL RELATIVISTIC THEORY

In special relativity, when $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ (in Cartesian coordinates), one does not use Einstein's equations, so that a homogeneous distribution of a perfect fluid in infinite flat space–time becomes admissible. We shall consider here the behavior of weak perturbations on the background of such a homogeneous field of a nonrotating perfect fluid. Then in the zeroth approximation $\tilde{G}$ coincides with the four-velocity of the fluid, $u = dt$ (in comoving coordinates; $t = x^0$), $J = 1$ (the background situation).

Now let a perturbation be introduced thus

$$\tilde{G}^\kappa = \delta_t^\kappa + \delta\tilde{G}^\kappa, \quad J = 1 + 2\delta\tilde{G}^t + \delta\tilde{G}^\kappa\delta\tilde{G}_\kappa \tag{8.1}$$

These relations might be considered as exact, though it is easy to see that, if one does not intend to consider the linear approximation only, it would be worth expressing $\delta\tilde{G}$ as a series of terms which describe all orders of magnitude of the perturbations. However, in the present context this will be of minor importance, and we shall deal with linear terms only. Then

$$L(J) = L(1) + 2\left[\frac{dL}{dJ}\right]_1 \delta\tilde{G}^t + \ldots \tag{8.2}$$

Here the dots denote higher order terms. The expression for $L(J)$ is equivalent (up to its sign) to the mass density, but one has still to take into account the field equations (4.3). These read, in similar notations,

$$\breve{\Phi}_{,\kappa} = \left[\frac{dL}{dJ}\right]_1 \delta_\kappa^t + \left[\frac{dL}{dJ}\delta_\kappa^\lambda + 2\frac{d^2L}{dJ^2}\delta_\kappa^t\delta_t^\lambda\right]_1 \delta\tilde{G}_\lambda + \ldots \tag{8.3}$$

The only property which matters in this expression is its gradient form. We arrive at the following two equations (the Latin indices being three-dimensional),

$$(\breve{\Phi})_{,t,i} = (\breve{\Phi})_{,t,i} \Rightarrow \left[\frac{dL}{dJ} + 2\frac{d^2L}{dJ^2}\right]_1 (\delta\tilde{G}_t)_{,i} = \left[\frac{dL}{dJ}\right]_1 (\delta\tilde{G}_i)_{,t} \tag{8.4}$$

and

$$(\tilde{\Phi})_{,i,j} = (\tilde{\Phi})_{,j,i} \Rightarrow \left[\frac{dL}{dJ}\right]_1 (\delta\tilde{G}_i)_{,j} = \left[\frac{dL}{dJ}\right]_1 (\delta\tilde{G}_j)_{,i} \tag{8.5}$$

This set of equations is satisfied if

$$\delta\tilde{G}_i = \left[\frac{dL/dJ + 2d^2L/dJ^2}{dL/dJ}\right]_1 \left(\int \delta\tilde{G}_t\, dt + \phi(\mathbf{x})\right)_{,i} \tag{8.6}$$

with two still undetermined functions, $\delta\tilde{G}_t(t, \mathbf{x})$ and $\phi(\mathbf{x})$. But we have not yet taken into account that $\delta\tilde{G}$ (as well as $\tilde{G}$) is divergenceless. This actually means that

$$\delta\tilde{G}^t_{,t} = -\delta\tilde{G}^i_{,t} = \delta\tilde{G}_{i,i} = \left[\frac{dL/dJ + 2d^2L/dJ^2}{dL/dJ}\right]_1 \Delta\left(\int \delta\tilde{G}_t\, dt + \phi(\mathbf{x})\right)$$

Δ is the Laplacian operator. Differentiating both sides of this relation with respect to $t = x^0$, we find at last

$$\frac{\partial^2\delta\tilde{G}_t}{\partial t^2} = \left[\frac{dL/dJ + 2d^2L/dJ^2}{dL/dJ}\right]_1 \Delta\delta\tilde{G}_t \tag{8.7}$$

which is a modification of the D'Alembert equation (involving a velocity different from that of light). Since the propagation properties of perturbations of the mass density μ, of the Lagrangian L, and of the field component $\tilde{G}_t$ mutually coincide in the first approximation, one concludes that the velocity of the low-amplitude density (sound) waves in a fluid is equal to

$$c_s = \sqrt{\left[\frac{dL/dJ + 2d^2L/dJ^2}{dL/dJ}\right]_1} \tag{8.8}$$

in units of the velocity of light. One has, of course, to remember that in this theory the laws of thermodynamics were used only implicitly (via equations of state). However, some important properties of the sound waves already can be seen in this result.

Let us consider first the simplest case, which is described by the equation of state (1.2) Then $L = -\sigma J^{\gamma/2}$, and we have

$$c_s = \sqrt{\gamma - 1} \tag{8.9}$$

When $\gamma = 1$, the perturbations do not propagate (in the comoving frame of the fluid); this is the case of an incoherent dust whose particles interact only gravitationally, *i.e.*, do not interact in a theory devoid of gravitation (special relativity). When $\gamma = 2$, we have stiff matter, in which (as is well known) sound propagates with the velocity of light, and this is exactly the case in our field-theoretic description: $c_s = 1$. When the value of γ lies between 1 and 2, we have more or less realistic fluids, the velocity of sound in them being less than that of light. For example, in the case of incoherent radiation (see a consideration of the Klein metric above), $c_s = 1/3$.

Turning to consideration of a polytrope (1.3) and taking into account its field-theoretic description (4.4), it is easy to find for the sound velocity (8.8) the corresponding form

$$c_s = \sqrt{\left[1 - 2\left(\frac{dJ}{dL}\right)^{-2}\frac{d^2J}{dL^2}\right]_1} \tag{8.10}$$

or, after a substitution of (4.4), exactly the standard expression

$$c_s = \sqrt{\gamma p/\mu} \tag{8.11}$$

It is worth stressing that in this section all considerations were restricted to the absence of a gravitational field as well as to weak perturbations of the fluid density, but the velocity of propagation of the perturbations may be relativistic. Thus the standard expression (8.11) represents in fact an exact generalization of c_s to the relativistic case; similarly, (8.9) gives the correct value of the velocity of sound in the ultrarelativistic cases important in the astrophysical context.

9. CONCLUDING REMARKS

As a summary of the results just described and in anticipation of some others (to be presented elsewhere), it is worth systematizing the present approach in the 3 + 1-dimensional spacetime. Our conclusions are essentially based on a consideration of the stress-energy tensor of r-form fields ($r = 0$, 1, 2, and 3), a fact which makes it clear why these conclusions partially coincide with those of Weinberg (1996, Section 8.8), where only the gauge covariance properties are taken into account.

A field whose potential is a skew-symmetric tensor of rank 4 (being identically a closed form in four dimensions) has only trivial field strength tensor, thus leaving for consideration the four fields used in (2.12).

The rank-3 field does not correspond to any real quantum particles (a result obtained in collaboration with H. Vargas Rodríguez, to be published elsewhere), thus these particles should be only virtual ones. In the classical theory, the rank-3 field with any degree of nonlinearity is equivalent to the appearance of the cosmological constant in Einstein's equations; when the Lagrangian density is proportional to $K^{1/2}$, the cosmological constant vanishes (thus suggesting a new interpretation of that very fact). The global nature of Mach's principle (admittedly related to rotation phenomena) also seems to justify consideration of the rank-3 field on a basis similar to that of the hypothetical fundamental cosmological field proposed by Sakurai (1960).

The rank-2 field describes (sometimes in interaction with the cosmological field) perfect fluids. The second quantization of the free rank-3 field yields real quanta, but they have only spin zero: all other particles appear as thoroughly virtual ones (another result in collaboration with H. Vargas Rodríguez, also not included in this paper).

Then comes the rank-1 field, which, in its linear case, is the Maxwellian one, making commentary unnecessary. And the last is the scalar field; I would add here (to the information given in Sections 2 and 3) only one comment on this field: its interaction with the rank-2 field mimics the electromagnetic field, thus exactly and with the same degree of simplicity reproducing, for example, the Reissner–Nordström black-hole spacetime without any electromagnetic field whatsoever (Mitskievich, 1998). This all follows from the stress-energy tensor (2.12).

It is worth mentioning that in the $2 + 1$-dimensional space–time the $r = 1$ field, formerly the (nonlinear) Maxwell one, now describes perfect fluids, while the $r = 2$ field is responsible for the cosmological term in the 3D Einstein equations.

The proposed description of perfect fluids is simple, and it yields exactly the same characteristics of perfect fluids and relations between these characteristics which are already well established in other approaches (see, *e.g.*, our consideration of the special relativistic limit of the theory, yielding the properties of sound waves in fluids). Moreover, our description suggests (and simplifies the realization of) some new lines for the generalization of the theory of perfect fluids (due to an extensive use of the Lagrangian formalism), in particular, it makes the second quantization of (the sound in) the perfect fluid in fact a mere routine.

The use of standard field-theoretic methods for the description of perfect fluids and their excitations (phonons) may also help in evaluating the effect of Cerenkov-type radiation of sound by narrow-fronted gravitational wave jets (or, gravitons) in matter. Another possible application of the proposed description of perfect fluids may be related to the construction of exact Einstein–Euler fields (gravitation and perfect fluid) using the properties of Killing–Yano tensors, if these would be admitted by the vacuum seed spacetimes (see the method proposed in Horský and Mitskievich (1989) for Einstein–Maxwell fields, which uses Killing vectors).

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